COMPLEX ANALYSIS

Sequence of Complex Nos:

— A set A of complex numbers is said to be a sequence if ∃ a mapping $f : N \longrightarrow A$ defined by

$$f(n) = z_n \, , \; \forall \; n \in N$$

i.e. $A = \{z_1, z_2 \cdots z_n \cdots\}$.

It is usually written as $\langle z_n \rangle$ or (z_n) or $\{z_n\}$

Limit of a sequence in C:

— A sequence $\langle z_n \rangle$ of complex numbers is said to have limit z_0 if for a given $\varepsilon > 0$, however small ∃ a +ve integer n_0 such that

$$|z_n - z_0| < \varepsilon \quad \forall \quad n \geq n_0 \, ?$$

The above definition can be abriviated or symbolized as

$$\lim_{n \to \infty} z_n = z_0$$

Convergent sequence:

— A sequence $\langle z_n \rangle$ is said to be convergent to the limit z_0, if for each $\varepsilon > 0$, however small ∃ a +ve integer n_0 such that

$$|z_n - z_0| < \varepsilon \quad \forall \; n \geq n_0 .$$

Cauchy's sequence:

— A sequence $\langle z_n \rangle$ of complex no. is said to be a Cauchy sequence if to every given $\varepsilon > 0$ $\exists$ a +ve integer n_0 such that

$$|z_n - z_m| < \varepsilon \quad \forall \; n, m \geq n_0$$

THEOREM:
Every Convergent sequence is a Cauchy sequence.

Proof:

Let $\langle z_n \rangle$ be a convergent sequence in C, converges (cges) to the limit $z_0 \in C$

$\therefore$ For a given $\varepsilon > 0$, however small $\exists$ a +ve integer $n_0 > 0$ such that

$$|z_n - z_0| < \frac{\varepsilon}{2} \quad \forall \; n \geq n_0 \quad \text{--- (1)}$$

Also, for the integer $m \geq n_0$ we have

$$|z_m - z_0| < \frac{\varepsilon}{2} \quad \forall \; m \geq n_0 \quad \text{--- (2)}$$

$$\because \; |z_n - z_m| = |(z_n - z_0) + (z_0 - z_m)|$$
$$\leq |z_n - z_0| + |z_0 - z_m|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

$\Rightarrow \langle z_n \rangle$ is a Cauchy sequence.

Since C is complete metric space. Therefore, every Cauchy sequence in C is convergent sequence in C.

Thus, a sequence $\langle z_n \rangle$ of complex number is convergent sequence iff it is a Cauchy sequence.

This is known as Cauchy's criterion of convergent sequences of complex numbers.

SERIES:

— An expression of the form $z_1+z_2+\dots+z_n+\dots = \sum_{n=1}^{\infty} z_n$ where, z_n is a term of the sequence $\langle z_n \rangle$

Now, if $S_n = z_1+z_2+\dots+z_n = \sum_{n=1}^{n} z_n$

Then, S_n is called nth partial sum of the series

CONVERGENT SERIES:

— A series $\sum_{n=1}^{\infty} z_n$ is said to be convergent if the sequence of its nth partial sum is convergent.

i.e. the sequence $\langle S_n \rangle$ is convergent

THEOREM:

A series $\sum z_n$ is convergent iff to every $\varepsilon > 0$ $\exists$ a +ve integer n_0 such that $|z_n + z_{n+1} + \dots + z_{n+p}| < \varepsilon \ \forall n \geq n_0$ and $\forall p \geq 0$

Proof:-

Let $S_n = z_1+z_2+\dots+z_n = \sum_{n=1}^{n} z_n$

Then, by Cauchy's general principle of convergence of sequence For a given $\varepsilon > 0$, $\exists$ a +ve integer n_0 such that $|S_n - S_m| < \varepsilon \ \forall n, m \geq n_0$

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$$\Rightarrow \left| \sum_{n=1}^{n+p} z_n - \sum_{m=1}^{m-1} z_m \right| < \varepsilon. \qquad p > 0.$$

$$\Rightarrow |z_1 + z_2 + \dots + z_{m-1} + z_n + z_{n+1} + z_{n+2} + \dots + z_{n+p} - (z_1 + z_2 + z_3 + \dots + z_{n-1})| < \varepsilon$$

$$\Rightarrow |z_n + z_{n+1} + \dots + z_{n+p}| < \varepsilon$$

If $p = 0$.
then $|z_n| < \varepsilon$.

$$\Rightarrow \lim_{n\to\infty} z_n = 0.$$

Thus, if the series $\sum z_n$ is convergent then $z_n \to 0$ as $n \to \infty$.

But the converse is not necessarily true.

UNIFORM CONVERGENCE OF SEQUENCE AND SERIES OF FUNCTIONS

— Let A be any arbitrary set and for each +ve integer n, let S_n be a function defined on the set A with values $S_n(x)$, then the sequence $\langle S_n(x) \rangle$ of complex value functions is said to be converges uniformly to $S(x)$ on the set A, if for each given $\varepsilon > 0$ $\exists$ a +ve integer n_0 such that

$$|S_n(x) - S(x)| < \varepsilon \quad \forall\ n \geq n_0.$$
$$\& \ \forall\ x \in A$$

THEOREM:

Cauchy's criterion for uniform convergence

The sequence $S_n(x)$ converges uniformly on a set $A \subseteq \mathbb{C}$ iff to each given $\varepsilon > 0$, $\exists$ a +ve integer n_0 such that

$$|S_n(x) - S_m(x)| < \varepsilon \quad \forall \; n, m \geq n_0 \;\&\; x \in A$$

Proof:

If Part:

Suppose the sequence $\langle S_n(x) \rangle$ converge uniformly to the limit $S(x)$.
Therefore, for a given $\varepsilon > 0$, $\exists$ a +ve integer n_0 such that

$$|S_n(x) - S(x)| < \frac{\varepsilon}{2} \quad \forall \; n \geq n_0$$

$\Rightarrow$ If $n \geq n_0$ then $|S_n(x) - S(x)| < \varepsilon/2$

Now, we have

$$|S_n(x) - S_m(x)| = |S_n(x) - S(x) + S(x) - S_m(x)|$$

$$\leq |S_n(x) - S(x)| + |S(x) - S_m(x)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

Thus,

$$|S_n(x) - S_m(x)| < \varepsilon \quad \forall \; m, n \geq n_0$$

Only Part:

Suppose the sequence $\langle S_n(x) \rangle$ in A is so defined that the equality

$|S_n(x) - S_m(x)| < \varepsilon \quad \forall\ m, n \geq n_0$ —— ①

hold in the set A

Then we have to prove that the sequence $\langle S_n(x) \rangle$ converges uniformly.

Since eqⁿ ① holds $\forall\ x \in A$

Therefore, by Cauchy's criterion of convergence for sequences keeping m fixed and taking limit as $n \to \infty$ we have

$|S_n(x) - S(x)| < \varepsilon \quad \forall\ n \geq n_0$

as limit $n \to \infty$

$\lim_{m \to \infty} S_m(x) = S(x)$.

$\Rightarrow \langle S_n(x) \rangle$ is uniformly ~~continuous~~. Convergent.

WEIRSTRASS'S M-TEST

The series $\sum f_n(x)$ of functions defined on a set A converges uniformly on A if

i) $|f_n(x)| \leq M_n$

For every +ve integer n and for every $x \in A$

where M_n is a +ve constant independent of x

2) The series $\sum M_n$ is convergent

Proof:-

Since, $\sum M_n$, each converges

Therefore, for a giv from Cauchy's criterion of converges of series for a given $\varepsilon > 0$, $\exists$ a +ve integer n_0 such that

$$|M_n| + |M_{n+1}| + \cdots + |M_{n+p}| < \varepsilon \quad \forall \; n \geq n_0$$

and $p \geq 0$

$$\Rightarrow M_n + M_{n+1} + \cdots + M_{n+p} < \varepsilon \quad \text{—— (1)}$$

$\forall \; n \geq n_0$ and $p \geq 0$.

$$\Rightarrow |f_n(x)| + |f_{n+1}(x)| + \cdots + |f_{n+p}(x)| < \varepsilon$$

$$\Rightarrow |f_n(x) + f_{n+1}(x) + \cdots + f_{n+p}(x)| < \varepsilon$$

$\Rightarrow \sum f_n(x)$ is uniformly convergent by Cauchy's criterion of converges of series.